

Reduction operators and exact solutions of generalized Burgers equations

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Reduction operators of generalized Burgers equations are studied. A connection between these equations and potential fast diffusion equations with power nonlinearity -1 via reduction operators is established. Exact solutions of generalized Burgers equations are constructed using this connection and known solutions of the constant-coefficient potential fast diffusion equation.

1 Introduction

In order to construct exact solutions of a partial differential equation, techniques involving non-classical symmetries (also called conditional symmetries, Q -conditional symmetries or reduction operators) are used, which leads to ansatzes reducing the initial equation to another one with a less number of independent variables. The nonclassical reduction method was proposed by Bluman and Cole [6, 7] and applied to many models of real-world phenomena, see [10, 18] and references therein. We use the name “reduction operator” for the notion of nonclassical symmetries, which is justified by results of [33].

The classical and generalized Burgers equations are used to model a wide variety of phenomena in physics, chemistry, mathematical biology, etc., see, e.g., [29, Chapter 4]. In the present paper we study the class of generalized Burgers equations of the form

$$u_t + uu_x + f(t, x)u_{xx} = 0 \quad (1)$$

where the arbitrary element f runs through the set of nonvanishing smooth functions of t and x . These equations were intensively investigated within the framework of symmetry analysis. The maximal Lie invariance group of the classical Burgers equation (with $f = 1$) was computed in [14]. It is the Burgers equation that was first considered from the nonclassical symmetry point of view after the seminal paper [7]. The corresponding results obtained in [30, 31] can be found in [1]. After a longtime break the interest to symmetry analysis of various generalizations of the Burgers equation was renewed. A point transformation between equations of the form (1) with $f = f(t)$ was detected in [8]. The study of form-preserving (admissible) transformations for class (1) by Kingston and Sophocleous [15] was a pioneer work on form-preserving transformations in the literature.

After [1, 30, 31], nonclassical symmetries of the Burgers equation were considered in a series of papers. In [23] solving the corresponding determining equations was partitioned into three cases, $\xi_u = 0$, $\xi_u = 1$ and $\xi_u = -\frac{1}{2}$, assuming $\tau = 1$ (cf. Section 2). It was proved that each nonclassical symmetry with $\xi_u = 0$ is equivalent to a Lie symmetry. The case $\xi_u = 1$ is rather simple and gives a single nonclassical symmetry. Particular solutions of the determining equations were found in the last case. Analogous results were presented in [2]. Nonclassical symmetries of a system of differential equations that is equivalent to the Burgers equation were studied in [18] in a similar way. The “no-go” case $\xi_u = -\frac{1}{2}$ was studied in [17] and then in [4].

Group classification of the subclass of the class (1) singled out by the constraint $f_x = 0$ was carried out in [28]. It was also shown that nontrivial nonclassical symmetries, which are not

equivalent to Lie symmetries, exist only for $f = \text{const}$, i.e. for the classical Burgers equation. Reduction operators of equations from the entire class (1) have not been exhaustively studied yet.

In the present paper we investigate reduction operators of generalized Burgers equations of the form (1) and establish their connection with solutions of potential fast diffusion equations with power nonlinearity -1 . Known exact solutions of the constant-coefficient potential fast diffusion equation allow us to construct parameterized families of exact solutions of certain generalized Burgers equations from the class (1).

The structure of this paper is as follows. The equivalence group of the class (1) is presented in the next section. Ibid the determining equations for the coefficients of reduction operators are derived and preliminary studied. Section 3 reveals the above connection with potential fast diffusion equations. In Section 4 we construct exact solutions of equations from the class (1).

2 Determining equations

The class (1) is normalized, i.e. any admissible transformation of this class is generated by a transformation from its equivalence group (see [11, 21] for precise definitions). The equivalence group G^\sim of the class (1) consists of the transformations

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\kappa x + \mu_1 t + \mu_0}{\gamma t + \delta}, \quad \tilde{u} = \frac{\kappa(\gamma t + \delta)u - \kappa\gamma x + \mu_1\delta - \mu_0\gamma}{\alpha\delta - \beta\gamma}, \quad \tilde{f} = \frac{\kappa^2}{\alpha\delta - \beta\gamma}f,$$

where $\alpha, \beta, \gamma, \delta, \mu_0, \mu_1$ and κ are constants; $\alpha, \beta, \gamma, \delta$ are defined up to a nonzero multiplier, $\alpha\delta - \beta\gamma \neq 0$ and $\kappa \neq 0$. The generalized equivalence group of the class (1) coincides with the usual one.

A *reduction operator* of an equation from the class (1) is a vector field of the form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u \quad (2)$$

that leads to an ansatz reducing the initial equation to an ordinary differential equation. We carry out the nonclassical symmetry analysis of the class (1) following [20, 22] and consider only reduction operators with $\tau \neq 0$ ($\tau = 0$ is the “no-go” case for evolution equations [11, 16, 32]). Due to the equivalence relation among reduction operators we can divide Q by τ , so claim that $\tau = 1$ in (2).

The conditional invariance criterion (see [33]) implies the system of determining equations with respect to the coefficients ξ and η . Partially solving this system, we derive expressions for ξ and η , which are polynomials of u with coefficients depending on t and x ,

$$\xi = \xi^1(t, x)u + \xi^0(t, x), \quad \eta = \frac{\xi^1(\xi^1 - 1)}{3f}u^3 + \left(\xi_x^1 + \frac{\xi^1\xi^0}{f}\right)u^2 + \eta^1(t, x)u + \eta^0(t, x).$$

Substituting these expressions into the system of determining equations and splitting with respect to u wherever it is convenient we obtain a system of differential equations where ξ^1, ξ^0, η^1 and η^0 are assumed as unknown functions,

$$\begin{aligned} \xi^1(\xi^1 - 1)(2\xi^1 + 1) &= 0, \\ -f_x(\xi^1)^2 + f_x\xi^1 + \xi^1\xi^0(2\xi^1 + 1) + 4f\xi^1\xi_x^1 &= 0, \\ \frac{f_t}{f}(\xi^1 - 1) - 2\frac{f_x}{f}\xi^1\xi^0 - \frac{f_x}{f}\xi^0 + 3f\xi_{xx}^1 + 2(\xi_x^1\xi^0 + \xi^1\xi_x^0) + (2\xi^1 + 1)\eta^1 - \xi_t^1 + \xi_x^0 &= 0, \\ \xi^0 + 2\xi^0\xi_x^0 + f\xi_{xx}^0 - \eta^0(2\xi^1 + 1) &= \frac{f_t}{f}\xi^0 + \frac{f_x}{f}(\xi^0)^2 + 2f\eta_x^1, \\ \eta_t + u\eta_x + f\eta_{xx} + 2\xi_x\eta - \frac{f_t}{f}\eta - \frac{f_x}{f}\xi\eta &= 0. \end{aligned}$$

We rewrite the last equation in terms of ξ^1 , ξ^0 , η^1 and η^0 and split it with respect to u only in particular cases.

The first determining equation obviously leads to the three possible cases depending on a value of ξ^1 : $\xi^1 = 1$, $\xi^1 = -\frac{1}{2}$ and $\xi^1 = 0$.

In the first case all reduction operator coefficients except ξ^1 vanish. The corresponding vector field $Q = \partial_t + u\partial_x$ is a unique common reduction operator for all equations from the class (1). The family of Q -invariant solutions consists of the functions $u(t, x) = (x + c_1)/(t + c_2)$, where c_1 and c_2 are arbitrary constants.

The second case is possible only if $f = 1 \bmod G^\sim$, i.e. we are faced with the problem of finding reduction operators with $\xi^1 = -\frac{1}{2}$ for the classical Burgers equation. It was established in [4, 17] that solving the corresponding system of three differential equations in three unknown functions ξ^0 , η^0 and η^1 is equivalent to solving three copies of the linear heat equation. Therefore, it has been referred to as a “no-go” problem.

Consider the third case where $\xi = \xi^0(t, x)$. It is known from [2] that for the classical Burgers equation each reduction operator of this kind is equivalent to a Lie symmetry operator. We prove that there are only two essential subcases for $f \neq \text{const}$. In the first one, when $\xi_{xx}^0 = 0$, the reduction operators have the form

$$Q = \partial_t + \frac{(at + b_1)x + d_1t + d_0}{at^2 + (b_1 + b_2)t + c} \partial_x + \frac{-(at + b_2)u + ax + d_1}{at^2 + (b_1 + b_2)t + c} \partial_u,$$

i.e. they are also equivalent to Lie symmetry operators. This paper is devoted to the other subcase, $\xi_{xx} \neq 0$. It can be proved that then $\eta_t^1 = (\eta^1)^2$ and $\eta^0 = -\eta^1 \xi^0$ and hence both η^1 and η^0 can be set to zero using a transformation from the equivalence group G^\sim .

We plan to present the proof and detailed consideration of all cases in a forthcoming paper.

3 Connection with potential fast diffusion equations

Consider reduction operators of an equation of the form (1) with $\xi_u = 0$ while $\xi_{xx} \neq 0$. As mentioned above, under these conditions we have $\eta = 0 \bmod G^\sim$ and the system of determining equations is reduced to the system

$$f_t + \xi f_x - \xi_x f = 0, \tag{3}$$

$$\xi_t + \xi \xi_x + f \xi_{xx} = 0, \tag{4}$$

which is well determined as it consists of two differential equations in two unknown functions $f(t, x)$ and $\xi(t, x)$. Moreover, this system is in Kovalevskaya form and hence it has no nontrivial differential consequences.

It is impossible to find the general solution of the system (3), (4). In other words, we single out a “no-go” case in which one cannot completely describe reduction operators of equations from the class (1). At the same time we are able to construct particular solutions of this system, which result in exact solutions of equations from the class (1) with special values of the arbitrary element f . The above construction is realized via establishing a connection between the system (3), (4) and a potential fast diffusion equation. For this purpose, we write equation (3) in conserved form, $(1/f)_t + (\xi/f)_x = 0$, and then introduce the corresponding potential $\theta = \theta(t, x)$, which is defined by the equations $\theta_t = \xi/f$ and $\theta_x = -1/f$. Hence f and ξ can be expressed in terms of derivatives of θ ,

$$f = -\frac{1}{\theta_x}, \quad \xi = -\frac{\theta_t}{\theta_x}. \tag{5}$$

Substituting these expressions into the equation (4) and factorizing, we obtain

$$(\theta_x \partial_t - \theta_t \partial_x) \left(\frac{\theta_t}{\theta_x} - \frac{\theta_{xx}}{(\theta_x)^2} \right) = 0.$$

Integrated once and multiplied by θ_x , the last equation leads to

$$\theta_t = \frac{\theta_{xx}}{\theta_x} + h(\theta) \theta_x, \quad (6)$$

where h is an arbitrary smooth function of θ . Thus, for an arbitrary solution θ of the equation (6) the vector field $Q^\theta = \partial_t - (\theta_t/\theta_x) \partial_x$ is a reduction operator of the equation of the form (1) with $f = -1/\theta_x$.

It is impossible to solve the equation (6) for an arbitrary h but some interesting results can be obtained when $h = \mu = \text{const}$. In this case the equation (6) is mapped by the transformation $\tilde{t} = t$, $\tilde{x} = x + \mu t$ of the independent variables to the potential fast diffusion equation

$$\theta_{\tilde{t}} = \frac{\theta_{\tilde{x}\tilde{x}}}{\theta_{\tilde{x}}}, \quad (7)$$

for which many exact solutions are known and all of them can be used in order to construct exact solutions of generalized Burgers equations.

As the Galilean boost $\tilde{t} = t$, $\tilde{x} = x + \mu t$, $\tilde{u} = u + \mu$, $\tilde{f} = f$, where μ is an arbitrary constant, belongs to the equivalence group G^\sim , we can assume that $\mu = 0 \bmod G^\sim$.

The general equation (6) is mapped to the variable-coefficient potential fast diffusion equation $\zeta_\tau = H(y) \zeta_{yy}/\zeta_y$ by a more complicated hodograph-type transformation

$$\begin{aligned} \text{the new independent variables:} \quad & \tau = t, \quad y = g(\theta), \\ \text{the new dependent variable:} \quad & \zeta = x, \end{aligned}$$

where $g(\theta) = \int e^{\int h d\theta} d\theta$ and $H(y)|_{y=g(\theta)} = e^{\int h d\theta}$. The corresponding fast diffusion equation is $v_\tau = (H(y) v_y/v)_y$. Variable-coefficient equations of the form $\zeta_\tau = H(y) \zeta_{yy}/\zeta_y$ are less studied than the constant-coefficient equation of the same form, i.e. equation (7). In particular, there are no exact solutions for these equations in the literature.

4 Exact solutions of generalized Burgers equations

The ansatz $u = \varphi(\omega)$, $\omega = \theta(t, x)$ constructed with the operator Q^θ reduces the generalized Burgers equation of the form (1) with the value $f = -1/\theta_x$ to the ordinary differential equation $\varphi_{\omega\omega} - \varphi\varphi_\omega - h(\omega)\varphi_\omega = 0$, which cannot be completely solved for a general value of h . At the same time, in the case $h = \mu = \text{const}$ the reduced equation is once integrated to the Riccati equation $\varphi_\omega = \frac{1}{2}\varphi^2 + \mu\varphi + 2\nu$, where ν is an integration constant. It is remarkable that this is the same case when the corresponding equation (6) for θ can be mapped to the potential fast diffusion equation (7). As the Riccati equation has constant coefficients it is obviously integrated using the substitution $\varphi = -2\psi_\omega/\psi$. Moreover, up to G^\sim -equivalence we can set $\mu = 0$ from the very beginning. As a result, we obtain the expression

$$\varphi = \begin{cases} -2\kappa \frac{c_1 e^{\kappa\omega} - c_2 e^{-\kappa\omega}}{c_1 e^{\kappa\omega} + c_2 e^{-\kappa\omega}}, & \nu < 0, \\ -\frac{2c_2}{c_1 + c_2\omega}, & \nu = 0, \\ 2k \frac{c_1 \sin k\omega - c_2 \cos k\omega}{c_1 \cos k\omega + c_2 \sin k\omega}, & \nu > 0, \end{cases} \quad (8)$$

where $\varkappa = \sqrt{-\nu}$, $k = \sqrt{\nu}$ and c_1 and c_2 are arbitrary constants (only the ratio of these constants is essential).

This allows one to construct, for each known solution of the potential fast diffusion equation (7), three families of solutions of the generalized Burgers equation with the value $f = -1/\theta_x$ by substituting $\omega = \theta(t, x)$ into (8). Note that in view of (4) the function $\xi = -\theta_t/\theta_x$ is also a solution of the same generalized Burgers equation with $f = -1/\theta_x$.

The widest list of solutions of the potential fast diffusion equation (7) is presented in [22]. They can be obtained, e.g., by integration of well-known solutions of the fast diffusion equation (see *ibid* and also [12, 19, 24, 25, 26]). These solutions together with the corresponding values of f and ξ are collected in Table 1.

The families of solutions constructed are extended to solutions of other generalized Burgers equations by transformations from the equivalence group G^\sim .

Table 1: Values of f , ξ and θ . Here λ is an arbitrary constant.

N	$f(t, x)$	$\xi(t, x)$	$\theta(t, x)$
1	$1 + e^{-t-x}$	e^{t+x}	$-\ln(e^t + e^{-x})$
2	-1	0	x
3	$1 - e^{-t-x}$	$-e^{t+x}$	$-\ln e^t - e^{-x} $
4	$-e^{-x}$	$-e^{-x}$	$e^x + t$
5	$t - x - \lambda t e^{-\frac{x}{t}}$	$1 - \lambda e^{-\frac{x}{t}}$	$\ln t + \int \frac{dw}{w - 1 + \lambda e^{-w}} \Big _{w=x/t}$
6	$-\frac{t^2 - x^2}{2t}$	$\frac{x}{t}$	$\ln \left \frac{x-t}{x+t} \right $
7	$-\frac{x^2}{2t}$	$\frac{x}{t}$	$-\frac{2t}{x}$
8	$-\frac{t^2 + x^2}{2t}$	$\frac{x}{t}$	$2 \arctan \frac{x}{t}$
9	$-\frac{\cos^2 x}{2t}$	$-\frac{\sin 2x}{2t}$	$2t \tan x$
10	$\frac{\cosh^2 x}{2t}$	$-\frac{\sinh 2x}{2t}$	$-2t \tanh x$
11	$-\frac{\sinh^2 x}{2t}$	$\frac{\sinh 2x}{2t}$	$-2t \coth x$
12	$\frac{\cos 2x - \cos 2t}{2 \sin 2t}$	$\frac{\sin 2x}{\sin 2t}$	$\ln \left \frac{\sin(x-t)}{\sin(x+t)} \right $
13	$\frac{\cosh 2t - \cosh 2x}{2 \sinh 2t}$	$\frac{\sinh 2x}{\sinh 2t}$	$\ln \left \frac{\sinh(x-t)}{\sinh(x+t)} \right $
14	$\frac{\sinh 2t - \sinh 2x}{2 \cosh 2t}$	$\frac{\cosh 2x}{\cosh 2t}$	$\ln \left \frac{\sinh(x-t)}{\cosh(x+t)} \right $
15	$\frac{\cosh 2t + \cosh 2x}{2 \sinh 2t}$	$-\frac{\sinh 2x}{\sinh 2t}$	$\ln \left \frac{\cosh(x-t)}{\cosh(x+t)} \right $
16	$\frac{\cos 2t - \cosh 2x}{2 \sin 2t}$	$\frac{\sinh 2x}{\sin 2t}$	$2 \arctan(\cot t \tanh x)$
17	$\frac{\cosh 2t - \cos 2x}{2 \sinh 2t}$	$\frac{\sin 2x}{\sinh 2t}$	$2 \arctan(\coth t \tan x)$

The values of θ in Cases 12–17 of Table 1 are non-Lie solutions of the equation (7). The maximal Lie invariance algebras of the corresponding generalized Burgers equations are zero, so it is clear that all exact solutions constructed for these equations are non-Lie ones. The other values of θ presented in the table are Lie solutions of the equation (7) but only ansatz (8) in Cases 2 and 6–8 and the values of ξ in all Cases 1–11 provide Lie solutions of the corresponding equations of the form (1).

5 Conclusion

In the present paper the reduction operators of the form (2) with $\tau \neq 0$ for the class (1) of generalized Burgers equations are completely classified and exact solutions for certain equations of this kind are constructed. (A more detail presentation of these and other results on symmetry analysis of generalized Burgers equations will appear soon.) As far as we know, there are only a few exhaustive descriptions of nonclassical symmetries for important classes of nonlinear differential equations parameterized by arbitrary functions in the literature. See, e.g., [9, 5] for heat equations with nonlinear source, [13] for generalized Huxley equation and [3] for systems of generalized Burgers equations. The classification of reduction operators of equations from the class (1) in the present paper gives one more example of such description.

The role of the equivalence group G^\sim is especially important in the course of this classification. Applying equivalence transformations essentially simplifies both the computation and final results.

Another specific feature of this classification is the appearance of a no-go case which includes all nontrivial reductions operators. There are no-go assertions on nonclassical symmetries of single differential equations [11, 16, 20, 32]. Similar results for classes of differential equations are less known. If in a particular case the corresponding system of determining equations on unknown functional parameters involved in coefficients of reduction operators and arbitrary elements of the class under consideration is not overdetermined, we are not able to find all values of arbitrary elements for which equations from the class possess nontrivial sets of reduction operators. At the same time, once a value of the tuple of arbitrary elements is fixed, the associated set of reduction operators may be completely defined. The above situation arises for generalized Burgers equations from the class (1). A vector field of the form (2) with $\tau = 1$ and $\eta = 0$ is a reduction operator of an equation of the class (1) if and only if the coefficient ξ and the arbitrary element f satisfy the well-determined system (3), (4). This obstacle was partially overcome by means of the discovered connection between this system and the potential fast diffusion equation (7). For each solution θ of the potential fast diffusion equation (7) we can construct at least three families of solutions of the generalized Burgers equation of the form (1) with $f = -1/\theta_x$.

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